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LETTER TO THE EDITOR

Constructing the N-soliton solution for the mKdV equation through constrained flows

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Abstract
Based on the factorization of soliton equations into two commuting integrable x- and t-constrained flows, we derive N-soliton solutions for a mKdV equation via its x- and t-constrained flows. We show that soliton solutions for soliton equations can be constructed directly from the constrained flows.

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1. Introduction

It is well known that there are several methods for deriving the N-soliton solution of soliton equations, such as the inverse scattering method, the Hirota method, the dressing method, the Darboux transformation, etc (see, e.g., [1–3] and references therein). In this letter, we propose a method of constructing N-soliton solutions for a mKdV equation directly through two commuting x- and t-constrained flows obtained from the factorization of the mKdV equation. It was shown in [4–7] that (1 + 1)-dimensional soliton equations can be factorized by x- and t-constrained flows, which can be transformed into two commuting x- and t-finite-dimensional integrable Hamiltonian systems. The Lax representation for constrained flows can be deduced from the adjoint representation of the auxiliary linear problem for soliton equations [8]. By means of the Lax representation and the standard method given in [9–11] we are able to introduce the separation variables for constrained flows [12–16] and to establish their Jacobi inversion problem [14–16]. Furthermore, the factorization of soliton equations and separability of the constrained flows allow us to find the Jacobi inversion problem for soliton equations [14–16]. By using the Jacobi inversion technique [17, 18], the N-gap solutions in terms of Riemann theta functions for soliton equations can be obtained, namely the constrained flows can be used to derive the N-gap solution. This letter shows that the x- and t-constrained flows and their Lax representation can also be used to directly construct the N-soliton solution for soliton equations. In fact, the method proposed in this letter, together with that in the previous paper [19], provides a general procedure to derive N-soliton solutions for soliton equations via their constrained flows.

2. The factorization of the mKdV hierarchy

We first briefly recall the constrained flows of the mKdV hierarchy and their Lax representation. The mKdV hierarchy

$$q_{t_{2n+1}} = Db_{2n+1} = D\frac{\delta H_{2n+1}}{\delta q} \qquad n = 0, 1, \ldots \tag{2.1}$$

with

$$H_{2n+1} = \frac{2a_{2n+2}}{2n+1}$$

is associated with the reduced AKNS spectral problem for $r = -q$ [1]:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = U\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \qquad U = \begin{pmatrix}-\lambda & q\\ -q & \lambda\end{pmatrix} \tag{2.2}$$

and the evolution equation of the eigenfunction

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_{t_{2n+1}} = V^{(2n+1)}(q,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{2.3}$$

where

$$V^{(2n+1)} = \sum_{j=0}^{2n+1}\begin{pmatrix}a_j & b_j\\ c_j & -a_j\end{pmatrix}\lambda^{2n+1-j} \tag{2.4}$$

with

$$a_0 = -1,\ b_0 = c_0 = a_1 = 0,\ b_1 = -c_1 = q,\ a_2 = -\tfrac{1}{2}q^2,\ b_2 = c_2 = -\tfrac{1}{2}q_x, \ldots$$

and, in general,

$$\begin{aligned} &b_{2m+1} = -c_{2m+1} = Lb_{2m-1} \qquad L = \tfrac{1}{4}D^2 + qD^{-1}qD \qquad D = \frac{\mathrm{d}}{\mathrm{d}x}\\ &DD^{-1} = D^{-1}D = 1 \qquad b_{2m} = c_{2m} = -\tfrac{1}{2}Db_{2m-1}\\ &a_{2m+1} = 0 \qquad a_{2m} = 2D^{-1}qb_{2m}. \end{aligned} \tag{2.5}$$

For the well known mKdV equation

$$q_t = Db_3 = \tfrac{1}{4}(q_{xxx} + 6q^2q_x) \tag{2.6}$$

$V^{(3)}$ is

$$V^{(3)} = \begin{pmatrix} -\lambda^3 - \frac{1}{2}q^2\lambda & q\lambda^2 - \frac{1}{2}q_x\lambda + \frac{1}{4}q_{xx} + \frac{1}{2}q^3\\ -q\lambda^2 - \frac{1}{2}q_x\lambda - \frac{1}{4}q_{xx} - \frac{1}{2}q^3 & \lambda^3 + \frac{1}{2}q^2\lambda \end{pmatrix}. \tag{2.7}$$

We have

$$\frac{\delta\lambda}{\delta q} = \psi_1^2 + \psi_2^2 \qquad L(\psi_1^2 + \psi_2^2) = \lambda^2(\psi_1^2 + \psi_2^2). \tag{2.8}$$

The x-constrained flows of the mKdV hierarchy consist of the equations obtained from the spectral problem (2.2) for N distinct real numbers λ_j and the restriction of the variational derivatives for the conserved quantities H_{2k_0+1} (for any fixed k_0) and λ_j defined by (see, e.g., [4–7, 20, 21])

$$\psi_{1j,x} = -\lambda_j\psi_{1j} + q\psi_{2j} \qquad \psi_{2j,x} = -q\psi_{1j} + \lambda_j\psi_{2j} \qquad j = 1, \ldots, N \tag{2.9a}$$

$$\frac{\delta H_{2k_0+1}}{\delta q} - \frac{1}{2}\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta q} \equiv b_{2k_0+1} - \frac{1}{2}\sum_{j=1}^{N}(\psi_{1j}^2 + \psi_{2j}^2) = 0. \tag{2.9b}$$

For $k_0 = 0$, (2.9b) gives

$$q = \tfrac{1}{2}(\langle\Psi_1, \Psi_1\rangle + \langle\Psi_2, \Psi_2\rangle) \tag{2.10}$$

where

$$\Psi_k = (\psi_{k1}, \ldots, \psi_{kN})^T \qquad k = 1, 2 \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N).$$

By substituting (2.10), (2.9a) becomes a finite-dimensional integrable Hamiltonian system (FDIHS):

$$\begin{aligned}
\Psi_{1x} &= -\Lambda\Psi_1 + \frac{1}{2}(\langle\Psi_1, \Psi_1\rangle + \langle\Psi_2, \Psi_2\rangle)\Psi_2 = \frac{\partial\overline{H}_0}{\partial\Psi_2}\\
\Psi_{2x} &= -\frac{1}{2}(\langle\Psi_1, \Psi_1\rangle + \langle\Psi_2, \Psi_2\rangle)\Psi_1 + \Lambda\Psi_2 = -\frac{\partial\overline{H}_0}{\partial\Psi_1}
\end{aligned} \tag{2.11}$$

with

$$\overline{H}_0 = -\langle\Lambda\Psi_1, \Psi_2\rangle + \tfrac{1}{8}(\langle\Psi_1, \Psi_1\rangle + \langle\Psi_2, \Psi_2\rangle)^2.$$

Under the constraint (2.10), the t-constrained flow obtained from (2.3) with $V^{(3)}$ given by (2.7) for N distinct λ_j can also be written as a FDIHS:

$$\Psi_{1t} = \frac{\partial\overline{H}_1}{\partial\Psi_2} \qquad \Psi_{2t} = -\frac{\partial\overline{H}_1}{\partial\Psi_1} \tag{2.12}$$

with

$$\begin{aligned}
\overline{H}_1 = {}& -\langle\Lambda^3\Psi_1, \Psi_2\rangle - \tfrac{1}{8}(\langle\Psi_1, \Psi_1\rangle + \langle\Psi_2, \Psi_2\rangle)^2\langle\Lambda\Psi_1, \Psi_2\rangle\\
&+\tfrac{1}{4}(\langle\Psi_1, \Psi_1\rangle + \langle\Psi_2, \Psi_2\rangle)(\langle\Lambda^2\Psi_1, \Psi_1\rangle + \langle\Lambda^2\Psi_2, \Psi_2\rangle) - \tfrac{1}{8}\langle\Lambda\Psi_1, \Psi_1\rangle^2\\
&-\tfrac{1}{8}\langle\Lambda\Psi_2, \Psi_2\rangle^2 + \tfrac{1}{4}\langle\Lambda\Psi_1, \Psi_1\rangle\langle\Lambda\Psi_2, \Psi_2\rangle + \tfrac{1}{128}(\langle\Psi_1, \Psi_1\rangle + \langle\Psi_2, \psi_2\rangle)^4.
\end{aligned}$$

The Lax representation for the constrained flows (2.11) and (2.12), which can be obtained from the adjoint representation of the Lax representation for the mKdV hierarchy [6, 8], is given by

$$M_x = [\tilde{U}, M] \qquad M_t = [\tilde{V}^{(3)}, M]$$

where $\tilde{U}$ and $\tilde{V}^{(3)}$ are obtained from U and $V^{(3)}$ by inserting (2.10) and the Lax matrix M is of the form

$$M = \begin{pmatrix} A(\lambda) & B(\lambda)\\ C(\lambda) & -A(\lambda)\end{pmatrix} \qquad A(\lambda) = -\lambda - \sum_{j=1}^{N}\frac{\lambda\lambda_j\psi_{1j}\psi_{2j}}{\lambda^2 - \lambda_j^2}$$

$$B(\lambda) = \frac{1}{2}(\langle\Psi_1, \Psi_1\rangle + \langle\Psi_2, \Psi_2\rangle) + \frac{1}{2}\sum_{j=1}^{N}\frac{\lambda_j}{\lambda^2 - \lambda_j^2}[(\lambda + \lambda_j)\psi_{1j}^2 - (\lambda - \lambda_j)\psi_{2j}^2]$$

$$C(\lambda) = -\frac{1}{2}(\langle\Psi_1, \Psi_1\rangle + \langle\Psi_2, \Psi_2\rangle) + \frac{1}{2}\sum_{j=1}^{N}\frac{\lambda_j}{\lambda^2 - \lambda_j^2}[(\lambda - \lambda_j)\psi_{1j}^2 - (\lambda + \lambda_j)\psi_{2j}^2].$$

The compatibility of (2.1)–(2.3) ensures that, if Ψ_1, Ψ_2 satisfy two commuting FDIHSs (2.11) and (2.12) simultaneously, then q given by (2.10) is a solution of the mKdV equation (2.6), namely the mKdV equation (2.6) is factorized by the x-constrained flow (2.11) and t-constrained flow (2.12).

3. Constructing the N-soliton solution for the mKdV equation

Hereafter we assume that $q(x, t)$, ψ_{1j}, ψ_{2j} be real functions. For a soliton solution we have $q(x, t) \to 0$, $\psi_{1j} \to 0$, $\psi_{2j} \to 0$, when $|x| \to \infty$. In order to obtain convenient formulae to construct N-soliton solutions, we need to rewrite all the formulae by using the complex version instead of the vector version. We let

$$\Phi = \Psi_1 + \mathrm{i}\Psi_2 \qquad \phi_j = \psi_{1j} + \mathrm{i}\psi_{2j}.$$

Then (2.11) and (2.12) become

$$\Phi_x = -\Lambda\Phi^* - \frac{\mathrm{i}}{2}\Phi^T\Phi^*\Phi \tag{3.1}$$

$$\Phi_t = -\Lambda^3\Phi^* - \frac{\mathrm{i}}{2}\Phi^T\Phi^*\Lambda^2\Phi + \frac{\mathrm{i}}{2}\Lambda\Phi^*\Phi^T\Lambda\Phi - \frac{\mathrm{i}}{2}\Phi\Phi^T\Lambda^2\Phi^* \tag{3.2}$$

where we have used $\overline{H}_0 = 0$.

The generating function of integrals of motion for the system (3.1) and (3.2), $\frac{1}{2}\operatorname{Tr} M^2(\lambda) = A^2(\lambda) + B(\lambda)C(\lambda)$, gives rise to

$$A^2(\lambda) + B(\lambda)C(\lambda) = \lambda^2 - 2\overline{H}_0 + \sum_{j=1}^{N}\frac{F_j}{\lambda^2 - \lambda_j^2}$$

where F_j, $j = 1, \ldots, N$, are N independent integrals of motion for the systems (3.1) and (3.2):

$$F_j = 2\lambda_j^3\psi_{1j}\psi_{2j} - \tfrac{1}{2}\Phi^T\Phi^*\lambda_j^2(\psi_{1j}^2 + \psi_{2j}^2) + \tfrac{1}{4}\lambda_j^2(\psi_{1j}^2 + \psi_{2j}^2)^2 + \tfrac{1}{2}\sum_{k\neq j}\frac{\lambda_j^2}{\lambda_j^2 - \lambda_k^2}P_{kj}$$

$$\begin{aligned}P_{kj} = {} & \lambda_j\lambda_k(4\psi_{1j}\psi_{2j}\psi_{1k}\psi_{2k} + \psi_{1j}^2\psi_{1k}^2 + \psi_{2j}^2\psi_{2k}^2 - \psi_{1j}^2\psi_{2k}^2 - \psi_{2j}^2\psi_{1k}^2) \\ & -\lambda_k^2(\psi_{1j}^2\psi_{1k}^2 + \psi_{2j}^2\psi_{2k}^2 + \psi_{1j}^2\psi_{2k}^2 + \psi_{2j}^2\psi_{1k}^2) \qquad j = 1, \ldots, N.\end{aligned}$$

Using (3.1), we have

$$\begin{aligned}&P_{kj} = -\tfrac{1}{2}[\lambda_k\phi_k\phi_j^*(\lambda_k\phi_k^*\phi_j - \lambda_j\phi_k\phi_j^*) + \lambda_k\phi_j\phi_k^*(\lambda_k\phi_j^*\phi_k - \lambda_j\phi_j\phi_k^*)] \\ &\lambda_j\phi_j\phi_j^*\partial_x^{-1}(\phi_j^2 + \phi^{*2}_j) = -(\phi_j\phi_j^*)^2 \\ &\lambda_k\phi_j\phi_k^* - \lambda_j\phi_k\phi_j^* = (\lambda_j^2 - \lambda_k^2)\partial_x^{-1}(\phi_j\phi_k).\end{aligned} \tag{3.3}$$

In a similar way to what we did in [19], in order to constructing N-soliton solutions, we have to set $F_j = 0$. By using (3.1) and (3.3) F_j can be rewritten as

$$\begin{aligned}F_j = {} & \frac{\mathrm{i}}{2}\lambda_j^2\phi_j^*\left[-\phi_{jx} + \frac{\mathrm{i}}{2}\sum_{k=1}^{N}\lambda_k\phi_k\partial_x^{-1}(\phi_j\phi_k)\right] \\ & -\frac{\mathrm{i}}{2}\lambda_j^2\phi_j\left[-\phi_{jx}^* - \frac{\mathrm{i}}{2}\sum_{k=1}^{N}\lambda_k\phi_k^*\partial_x^{-1}(\phi_j^*\phi_k^*)\right] = 0\end{aligned}$$

which leads to

$$\phi_{jx} = -\gamma_j\phi_j + \frac{\mathrm{i}}{2}\sum_{k=1}^{N}\lambda_k\phi_k\partial_x^{-1}(\phi_j\phi_k) \qquad j = 1, \ldots, N$$

or equivalently

$$\Phi_x = -\Gamma\Phi + \frac{\mathrm{i}}{2}\partial_x^{-1}(\Phi\Phi^T)\Lambda\Phi = -\Gamma\Phi + R\Phi \tag{3.4}$$

where $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_N)$, γ_j are undetermined real numbers and

$$R = \frac{\mathrm{i}}{2}\partial_x^{-1}(\Phi\Phi^T)\Lambda. \tag{3.5}$$

Notice that

$$\frac{\mathrm{i}}{2}\Phi\Phi^T = R_x\Lambda^{-1} \qquad \Lambda R = R^T\Lambda. \tag{3.6}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} R_x &= \frac{\mathrm{i}}{2}\partial_x^{-1}(\Phi_x\Phi^T + \Phi\Phi_x^T)\Lambda \\ &= \partial_x^{-1}(-\Gamma R_x + RR_x - R_x\Gamma + R_x R) = -\Gamma R - R\Gamma + R^2. \end{aligned} \tag{3.7}$$

We now show that $\Gamma = \Lambda$. In fact, it is found from (3.4) and (3.7) that

$$\begin{aligned} \Phi_{xx} &= -\Gamma\Phi_x + R\Phi_x + R_x\Phi \\ &= -\Gamma(-\Gamma\Phi + R\Phi) + R(-\Gamma\Phi + R\Phi) + (-\Gamma R - R\Gamma + R^2)\Phi \\ &= \Gamma^2\Phi + 2R_x\Phi = \Gamma^2\Phi + \mathrm{i}\Phi\Phi^T\Lambda\Phi. \end{aligned}$$

On the other hand (3.1) yields

$$\Phi_{xx} = \Lambda^2\Phi + \mathrm{i}\Phi\Phi^T\Lambda\Phi$$

which implies $\Gamma = \Lambda$. Therefore we have

$$\Phi_x = -\Lambda\Phi + R\Phi \tag{3.8}$$

$$R_x = \frac{\mathrm{i}}{2}\Phi\Phi^T\Lambda = -\Lambda R - R\Lambda + R^2. \tag{3.9}$$

To solve (3.8), we first consider the linear system

$$\Psi_x = -\Lambda\Psi.$$

It is easy to see that

$$\Psi = (\alpha_1(t)\mathrm{e}^{-\lambda_1 x}, \ldots, \alpha_N(t)\mathrm{e}^{-\lambda_N x})^T.$$

Take the solution of (3.8) to be of the form

$$\Phi = (I - M)\Psi. \tag{3.10}$$

Then M has to satisfy

$$M_x = M\Lambda - \Lambda M - R + RM. \tag{3.11}$$

Comparing (3.11) with (3.9) one finds

$$M = \frac{1}{2}R\Lambda^{-1} = \frac{\mathrm{i}}{4}\partial_x^{-1}(\Phi\Phi^T). \tag{3.12}$$

Equation (3.10) implies that

$$\Psi = \sum_{n=0}^{\infty} M^n\Phi. \tag{3.13}$$

By using (3.12) and (3.13), it is found that

$$\begin{aligned} \frac{\mathrm{i}}{4}\partial_x^{-1}(\Psi\Psi^T) &= \frac{\mathrm{i}}{4}\partial_x^{-1}\sum_{n=0}^{\infty}\sum_{l=0}^{n} M^l\Phi\Phi^T M^{n-l} \\ &= \partial_x^{-1}\sum_{n=0}^{\infty}\sum_{l=0}^{n} M^l M_x M^{n-l} = \sum_{n=1}^{\infty} M^n. \end{aligned}$$

Setting

$$V = (V_{kj}) = \frac{\mathrm{i}}{4}\partial_x^{-1}(\Psi\Psi^T) \qquad V_{kj} = -\frac{\mathrm{i}}{4}\frac{\alpha_k(t)\alpha_j(t)}{\lambda_k + \lambda_j}\mathrm{e}^{-(\lambda_k+\lambda_j)x}$$

one obtains

$$(I+V)\Phi = \Psi \qquad \text{or} \qquad \Phi = (I-M)\Psi = (I+V)^{-1}\Psi. \tag{3.14}$$

Notice that (3.1) and (3.8) give rise to

$$\Lambda\Phi^* = \left(\Lambda - R - \frac{\mathrm{i}}{2}q\right)\Phi. \tag{3.15}$$

By inserting (3.9) and (3.15), (3.2) reduces to

$$\begin{aligned}\Phi_t = \Bigg[& -\Lambda^2\left(\Lambda - R - \frac{\mathrm{i}}{2}q\right) - \frac{\mathrm{i}}{2}q\Lambda^2 + \left(\Lambda - R - \frac{\mathrm{i}}{2}q\right)(-\Lambda R - R\Lambda + R^2) \\ & -(-\Lambda R - R\Lambda + R^2)\left(\Lambda - R - \frac{\mathrm{i}}{2}q\right)\Bigg]\Phi = -\Lambda^3\Phi + R\Lambda^2\Phi.\end{aligned} \tag{3.16}$$

Let Ψ satisfy the linear system

$$\Psi_t = -\Lambda^3\Psi. \tag{3.17}$$

Then

$$\Psi = (\alpha_1(t)\mathrm{e}^{-\lambda_1 x}, \ldots, \alpha_N(t)\mathrm{e}^{-\lambda_N x})^T \qquad \alpha_i(t) = \beta_j \mathrm{e}^{-\lambda_j^3 t} \quad j = 1, \ldots, N. \tag{3.18}$$

We now show that Φ determined by (3.14) and (3.18) satisfies (3.16). In fact, we have

$$\begin{aligned}\Phi_t &= -(I+V)^{-1}\frac{\mathrm{i}}{4}\partial_x^{-1}(\Psi_t\Psi^T + \Psi\Psi_t^T)(I+V)^{-1}\Psi + (I+V)^{-1}\Psi_t \\ &= (1-M)(\Lambda^3 V + V\Lambda^3)\Phi - (1-M)\Lambda^3(1+V)\Phi \\ &= -\Lambda^3\Phi + (I-M)V\Lambda^3\Phi + M\Lambda^3\Phi \\ &= -\Lambda^3\Phi + 2M\Lambda^3\Phi = -\Lambda^3\Phi + R\Lambda^2\Phi.\end{aligned}$$

Therefore Φ given by (3.14) and (3.18) satisfies (3.1) and (3.2) simultaneously, and $q = \Phi^T\Phi^*$ is the solution of the mKdV equation (2.6). Notice that

$$\begin{aligned}\partial_x(\Psi^T\Phi) &= -\Psi^T\Lambda\Phi + \Psi^T(-\Lambda + R)\Phi \\ &= \Psi^T(-2I + 2M)\Lambda\Phi = -2\Phi^T\Lambda\Phi \\ q_x = \frac{1}{2}(\Phi_x^T\Phi^* + \Phi^T\Phi_x^*) &= \frac{1}{2}\left[\left(-\Phi^{*T}\Lambda - \frac{\mathrm{i}}{2}q\Phi^T\right)\Phi^* + \Phi^T\left(-\Lambda\Phi + \frac{\mathrm{i}}{2}q\Phi^*\right)\right] \\ &= -\tfrac{1}{2}(\Phi^{*T}\Lambda\Phi^* + \Phi^T\Lambda\Phi) = -\mathrm{Re}\,(\Phi^T\Lambda\Phi).\end{aligned}$$

So we have

$$q = \tfrac{1}{2}\,\mathrm{Re}\,(\Psi^T\Phi) = \tfrac{1}{2}\,\mathrm{Re}\sum_{k=1}^{N}\alpha_k(t)\mathrm{e}^{-\lambda_k x}\phi_k. \tag{3.19}$$

Finally, as pointed out in [1], formulae (3.14) and (3.19) give rise to the well known N-soliton solution of a mKdV equation (2.6)

$$u = 2\partial_x \mathrm{Im}\ln(\det(I+V)).$$

4. Conclusion

We first factorize the mKdV equation into two commuting integrable x- and t-constrained flows, then use them and their Lax representation to directly derive the N-soliton solution for a mKdV equation. The method proposed in the present letter and a previous paper [19] provides a general procedure for constructing N-soliton solutions for soliton equations via their x- and t-constrained flows and can be applied to other soliton equations.

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